The Game of Nim

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In the world of uncertainty today, it is always a pleasure to find something which is quite definite and certain. In games of chance the odds always seem to be against us; and if possibly they might be in favor of us, we seem to lose anyway. However, there is a game that, while it appears to be a very complex game of chance and perhaps some skill, in reality has a simple and complete mathematical theory. For the person understanding this theory, it is virtually impossible to lose. I say virtually because there is one very trivial situation in which it would be possible to lose while knowing the theory. This situation will be discussed later.

Such a game with a complete mathematical theory is the game of Nim. The game was not invented by a Chinese gentleman called Nim as you may have thought from the title. In fact no one knows when or where the game was originated or who first thought of it. The name was proposed about 1900 by Charles L. Bouton, who is responsible for the development of the theory of the game, principally because no one else had taken the trouble to name it before then. The name he proposed was Nim, which means to take or to filch. It will be seen that this name is appropriate as the description of the game is given.

This game is for two players, who shall be referred to as A and B. A number of objects (matches, coins, or chips will do), which we will call units, is divided into three piles of arbitrary size. It is best, however, to agree beforehand that no two piles will be the same at the beginning of the game. The first player, A, then selects one of the piles and draws as many units as he wishes from that pile or takes the whole pile if he wants. Then B takes his turn and draws as many units as he wishes from one of the piles. The play continues in this manner, A and B playing alternately, until the player picking up the last unit or units, as the case may be, is declared the winner. The only restriction on play is that each player, when his turn comes, must take at least one unit and is restricted to drawing from one pile only.

It will be shown that if one of the players, say A, leaves one of a certain set of numbers (i.e., leaves a special arrangement of numbers in each pile) and after that plays in a certain manner without making a mistake, it will be impossible for B to win. We shall call any of these special arrangements a safe combination, which is also a term set forth by Bouton. The proof will show then, that if A leaves a safe combination and regardless of what B may draw, A can leave a different safe combination. The play will continue in this way, A leaving a safe combination and B leaving an unsafe combination (defined obviously as a combination which is not safe) until A is the eventual winner.

The proof will proceed in a rather backward manner. First, it will consist of showing how to determine a safe combination; then, it will show that this method will always determine a safe combination and that it is possible to determine all possible safe combinations by this method. First, since much of the proof which follows depends on the binary scale of notation, it would be well to give a description of it. This scale of notation depends upon the sum of the different powers of the number 2. A few examples will best describe this. The number 10 will appear as follows— 1010, which means $1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$. The number 23 will be $10111 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$. So we see that the binary scale of notation is the coefficients from right to left of the ascending powers of 2 beginning with the zero power.

A safe combination is determined in the following manner: write the number of units in each pile in the binary scale of notation, and place these numbers in three horizontal lines so that the coefficients of the zero power of 2 of each number is in the same vertical column. If the sum of each vertical column is then zero or two (i.e., congruent to 0, mod. 2), the set of numbers is a safe combination. For example

$$\begin{array}{r} 110 \\ 1000 \\ 1110 \end{array}$$

or 6, 8, 14 is a safe combination. From the arrangement of the numbers in the columns it can be seen at once that any two numbers automatically determine the third which will give a safe combination. From this it follows that if a, b, c forms a safe combination, any two of the numbers determine the remaining one, that is, the system is closed. From the manner in which the safe combination is determined it becomes obvious that a special form of a safe combination will be where two piles are the same and the third is zero.

Theorem I: If A leaves a safe combination on the table, B cannot leave a safe combination on his next move.

Proof: B must change one and only one pile. We have already stated that any two of the piles determine the third pile of a safe combination. But, since A left a safe combination (by hypothesis), he also left the number so determined in the third pile (i.e., the pile

THE NIMATRON is a machine which will play the game of Nim against a human opponent. Invented in 1939 by Dr. E. V. Condon, G. L. Tawney, and W. A. Derr of Westinghouse Electric Corp., it has four rows of seven lights, different numbers of these lights being lit for different game set-ups. The player can extinguish one or more lights in any one row, and then the machine plays. If the player makes one mistake, he cannot win. Originally designed for use at the New York World's Fair, the Nimatron has been at the Buhl Planetarium in Pittsburgh since 1942. (*Picture courtesy Westinghouse Elect. Corp.*)

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from which B draws); and B cannot leave that number. Therefore, it is impossible for B to leave a safe combination.

Theorem II: *If A leaves a safe combination and B diminishes one of the piles, A can always diminish one of the two remaining piles and leave a safe combination.* Proof: In order to prove this theorem, let the numbers in each pile be written in the binary scale of notation with the coefficients of each zero power in the same vertical column. When B diminishes one one of the piles, the first change which occurs in the number in going from left to right is that one of the 1's is changed to a 0. This must be true; if one of the 0's were changed to a 1, the number would be larger regardless of the changes made in the remaining digits to the right. This is true because the number 100.....(n ciphers) or 2^n is greater than the number 111....(n ones), or $2^{n-1}+2^{n-2}+....2+1=2^n-1$. Then if we take the first column from the left in which a change occurs, one and only one of the two remaining rows can have a one in the same column. This must be true since A had left a safe combination: in a safe combination all the columns must add up to either zero or two. Since the first change which occurs when B diminishes one of the piles is a 1 changing to a 0, there must be another one in the same column but a different row. Now if A chooses this row with the single 1 in this column and changes the 1 to a zero and proceeds to the right changing the 0's to 1's or 1's to 0's as is needed, he can again leave a safe combination. The columns to the left of the one in which the original change took place remain unchanged and still fit the required form for a safe combination. The

Continued on Page 38